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Quasi-stationary Approximation for Reaction–Diffusion Systems

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Among chemical interactions describable as nonlinear reaction–diffusion systems, the simplest mathematically is the formation of a polymer. If $u(x, t)$ denotes concentration of the monomer, then irreversible formation of a polymer composed of n monomers is described by

$$-u_t + \alpha\, \Delta u = nk_1 u^n,$$

where α is the diffusion rate of monomer and k_1 is the reation rate. In the case of a similar but reversible reaction, let v denote concentration of the polymer; then

$$-u_t + \alpha\, \Delta u = nk_1 u^n - nk_2 v,$$
$$-v_t + \beta\, \Delta v = -k_1 u^n + k_2 v,$$

where k_2 is the reaction rate for breakdown of the polymer. Here β, the diffusion rate of polymer, will in general be less than α. Of a similar order of complexity is the irreversible reaction of two substances to form a third; this is described by

$$-u_t + \alpha\, \Delta u = nk_1 u^n v^m,$$
$$-v_t + \beta\, \Delta v = mk_1 u^n v^m,$$

where u and v are now the concentrations of the two reactants.

Positive solutions of the first and last of these reaction schemes have been studied by Kahane [3, 4], who shows that, if the Dirichlet boundary data have suitable limits as $t \to \infty$, then the solutions approach those of the corresponding elliptic problems obtained by dropping the time-derivative terms and using the limiting form of the data.

There is another circumstance in which, intuitively, a suitable approximation is provided by an elliptic problem: the case of data changing only slowly with time. Consider, for example, the first equation above, and

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suppose that the time derivative of the boundary data tends to zero as $t \to \infty$. Then since the u_t term "ought" to be small, it is reasonable to hope that the solution in the domain ω of

$$\alpha \Delta U = nk_1 U^n, \qquad U|_{\partial\omega} = u|_{\partial\omega}$$

will there approximate u as t gets large. Note that in this *quasi-stationary* approximation t plays the role of a parameter.

Our goal here is to show the validity of the quasi-stationary approximation for positive solutions of the three reaction–diffusion systems described above. The following notation will be used. ω will denote a bounded subset of Euclidean n-space R^n, with smooth boundary $\partial\omega$. Let $\Omega_T \equiv \omega \times [0, T]$, $\Omega \equiv \omega \times [0, \infty)$, $\Gamma_T \equiv (\omega \times \{0\}) \cup (\partial\omega \times [0, T))$, $\Gamma \equiv (\omega \times \{0\}) \cup (\partial\omega \times [0, \infty))$. Denote by $C^{2,1}(\Omega)$ the set of all functions in $n+1$ variables defined and continuous on Ω and continuously differentiable twice with respect to the first n independent variables x and once with respect to the $(n+1)$st variable t in the interior of Ω.

The single equation will be treated first, as the results established there are needed in the other cases.

1. Irreversible Polymer Formation

Since it requires no greater effort, we shall consider a generalization of the single nonlinear reaction–diffusion equation introduced above. Let L denote the uniformly elliptic operator defined by

$$Lu = \sum_{j,k=1}^{n} a_{jk}(x)\, u_{x_j} u_{x_k} + \sum_{j=1}^{n} a_j(x)\, u_{x_j} - a_0(x)\, u;$$

here the coefficients are Hölder continuous in $\bar{\omega}$, $a_0 \geqslant 0$, and there exists a constant $a > 0$ such that $(\xi_1, \xi_2, \ldots, \xi_n) \in R^n$ implies that

$$\sum_{j,k=1}^{n} a_{jk}(x)\, \xi_j \xi_k \geqslant a \sum_{j=1}^{n} \xi_j^2$$

for $x \in \bar{\omega}$. Let $f(x, t, u)$ be continuously differentiable with $f_u \geqslant 0$; let $\phi(x, t)$ be continuously differentiable in t and nonnegative. We consider positive $C^{2,1}(\Omega)$ solutions of the nonlinear reaction–diffusion equation

$$-u_t + Lu = f(x, t, u), \qquad u|_{\Gamma} = \phi; \tag{1}$$

the existence of a unique positive solution is shown by Kahane [4]. If

$f(x, t, u) \to \bar{f}(x, u)$ and $\phi(x, t) \to \bar{\phi}(x)$ as $t \to \infty$, then Kahane shows that $u \to U$ as $t \to \infty$ provided the Dirichlet problem

$$Lv = \bar{f}(x, v), \qquad v|_{\partial\omega} = \bar{\phi}$$

has the unique solution U. As discussed previously, it is also reasonable to expect this sort of behavior under a different circumstance: if $f_t(x, t, u) \to 0$, $\phi_t(x, t) \to 0$ as $t \to \infty$, corresponding to ever more slowly varying forcing function and boundary conditions, then the solution of (1) should approach, as $t \to \infty$, the solution of the Dirichlet problem

$$LU = f(x, t, U), \qquad U|_{\partial\omega} = \phi, \tag{2}$$

in which t plays the role of a parameter. We show that this is indeed the case.

Our primary tool will be the construction of suitable *majorant* or *barrier* functions, as described in the following lemma:

LEMMA 1. *Let $\Phi(x, t)$, $\Psi(x, t) \in C^{2,1}(\Omega_T)$ and satisfy*

$$|-\Phi_t + L\Phi| \leqslant \Psi_t - L\Psi$$

in the interior of Ω_T. If also

$$|\Phi| \leqslant \Psi$$

on Γ_T, then

$$|\Phi| \leqslant \Psi$$

throughout Ω_T.

For obvious reasons, Ψ is called a majorant for Φ. The lemma itself is a straightforward consequence of the maximum principle; an easily modified proof for elliptic rather than parabolic operators can be found in Eckhaus and de Jaeger [2].

LEMMA 2. *Let $U(x, t)$ be a twice continuously differentiable (in x) solution of* (2); *then $U_t(x, t)$ is a continuous function satisfying*

$$LU_t = f_t(x, t, U) + f_u(x, t, U)\, U_t, \qquad U_t|_{\partial\omega} = \phi_t.$$

Proof. Using the consequence of the maximum principle mentioned below it is easy to prove that $U(x, t)$ is continuous in t uniformly in $x \in \omega$. To prove differentiability, set

$$Q(x, t, h) = [U(x, t + h) - U(x, t)]/h$$

and let $w(x, t)$ be the solution of the linear Dirichlet problem

$$Lw - f_u(x, t, U(x, t))\, w = f_t(x, t, U(x, t)), \qquad w|_{\partial\omega} = \phi_t(x, t).$$

Then $Q - w$ satisfies

$$\begin{aligned} L[Q-w] &- f_u(x, t, U^*)[Q-w] \\ &= [f_u(x, t, U^*) - f_u(x, t, U(x, t))]\, w \\ &\quad + [f_t(x, t^*, U(x, t+h)) - f_t(x, t, U(x, t))] \\ &\equiv g(x, t, h), \end{aligned}$$

$$[Q-w]|_{\partial\omega} = \frac{\phi(x, t+h) - \phi(x, t)}{h} - \phi_t(x, t) \equiv m(x, t, h),$$

where t^* lies between t and $t+h$ and U^* lies between $U(x, t)$ and $U(x, t+h)$. Since $f_u \geqslant 0$, by a well-known consequence of the maximum principle [1, p. 153] we have that

$$|Q - w| \leqslant k(\max_{x\in\omega} |g(x, t, h)| + \max_{x\in\partial\omega} |m(x, t, h)|)$$

for a constant k depending only on L and ω. Since f_t, f_u are continuous, it follows easily that $g(x, t, h) \to 0$ uniformly in $x \in \omega$ as $h \to 0$. Since for some $\hat{t}$ between t and $t+h$

$$m(x, t, h) = \phi_t(x, \hat{t}) - \phi_t(x, t)$$

by the mean value theorem, the convergence of $m(x, t, h)$ to zero with h uniformly in $x \in \partial\omega$ follows from an elementary topological argument and the continuity of ϕ_t in *both* variables. The lemma follows.

LEMMA 3. *In addition to our standing smoothness hypotheses, let*

(i) $f(x, t, 0)$ *and* $\phi(x, t)$ *be bounded on* Ω *and* $\partial\omega \times [0, \infty)$, *respectively*;

(ii) $f_t(x, t, u) \to 0$ *as* $t \to \infty$ *uniformly in* x, u *for* $x \in \omega$ *and* $0 \leqslant u \leqslant N$ (*every* $N > 0$); *and*

(iii) $\phi_t(x, t) \to 0$ *uniformly in* $x \in \partial\omega$ *as* $t \to \infty$.

Let U *be a nonnegative solution of the Dirichlet problem* (2). *Then* $U_t(x, t) \to 0$ *as* $t \to \infty$, *uniformly in* $x \in \omega$.

Proof. Writing

$$LU = k(x, t, U)\, U + f(x, t, 0)$$
$$\equiv \frac{f(x, t, U) - f(x, t, 0)}{U}\, U + f(x, t, 0)$$

(where k is understood to be defined by continuity as $f_u(x, t, 0)$ when $U = 0$) in the form

$$LU - k(x, t, U)\, U = f(x, t, 0),$$

we get from the monotonicity of f that $k \geqslant 0$. Hence by the maximum principle, we get that

$$|U(x, t)| \leqslant \text{const}(\sup_{\Omega} |f(x, t, 0)| + \sup_{x \in \partial\omega, t \geqslant 0} |\phi(x, t)|).$$

That is, $U(x, t)$ is bounded, say by N, for all $t \geqslant 0$.

By Lemma 2, U_t satisfies

$$LU_t - f_u(x, t, U)\, U_t = f_t(x, t, U), \qquad U_t|_{\partial\omega} = \phi_t(x, t).$$

Again by the maximum principle we have that

$$|U_t(x, t)| \leqslant \text{const}(\sup_{x \in \omega} \sup_{0 \leqslant U \leqslant N} |f_t(x, t, U)| + \sup_{x \in \partial\omega} |\phi_t(x, t)|),$$

which tends to zero as $t \to \infty$, uniformly in $x \in \omega$.

THEOREM 1. *In addition to our standing smoothness hypotheses, let*

(i) $f(x, t, 0)$ *and* $\phi(x, t)$ *be bounded on* Ω *and* $\partial\omega \times [0, \infty)$, *respectively*;

(ii) $f_t(x, t, u) \to 0$ *as* $t \to \infty$ *uniformly in* x, u *for* $x \in \omega$ *and* $0 \leqslant u \leqslant N$ (*every* $N > 0$);

(iii) $f_u(x, t, u)$ *be bounded on* $\omega \times [0, N]$ *for every* $N > 0$;

(iv) $\phi_t(x, t) \to 0$ *uniformly in* $x \in \partial\omega$ *as* $t \to \infty$.

Let U be the unique nonnegative solution of the Dirichlet problem (2). *Then*

$$u(x, t) - U(x, t) \to 0$$

uniformly in $x \in \omega$ *as* $t \to \infty$.

Proof. We have by the corollary above that

$$-U_t + LU = f(x, t, U) - \eta,$$

where $\eta \equiv U_t$ tends to zero as $t \to \infty$, uniformly in $x \in \omega$. Given $\varepsilon > 0$, choose T so large that $t \geqslant T$ implies $|\eta| \leqslant \varepsilon$. Let $W \equiv u - U$; then

$$-W_t + LW = f(x, t, u) - f(x, t, U) + \eta \equiv g(x, t, u, W)\, W + \eta,$$
$$W|_\Gamma = 0,$$

where

$$g(x, t, u, u - U) \equiv \begin{cases} \dfrac{f(x, t, u) - f(x, t, U)}{u - U}, & u \neq U, \\ f_u(x, t, u), & u = U, \end{cases}$$
$$\equiv f_u(x, t, u^*) > 0$$

for some $u^*(x, t)$ between u and U. Thus g is a bounded positive function. We have

$$-W_t + LW - gW = \eta,$$
$$W|_{\partial\omega \times [0,\infty)} = 0,$$
$$W|_{\omega \times \{0\}} = \phi(x) - U(x, 0).$$

Let W^1 be the solution of $-W^1_t + LW^1 - gW^1 = \eta$, $W^1|_{\partial\omega \times [0,\infty)} = 0$, $W^1|_{\omega \times \{0\}} = 0$, and W^2 the solution of $-W^2_t + LW^2 - gW^2 = 0$, $W^2|_{\partial\omega \times [0,\infty)} = 0$, $W^2|_{\omega \times \{0\}} = \phi(x) - U(x, 0)$. Then $W = W^1 + W^2$ and it suffices to show that both W^1 and W^2 tend to zero as $t \to \infty$.

Regarding W^2 we show that for appropriate choice of positive constants α, β, and M there is a barrier function for W^2 of the form

$$\Psi = Me^{-\alpha t}\left[1 - \exp\left(-\beta \sum_{i=1}^{n} (x_i + 2l_i)\right)\right];$$

here the l_i are such that $\omega \subset \{x : |x_i| \leqslant l_i\}$. Indeed we have

$$\begin{aligned} \Psi_t - L\Psi \geqslant Me^{-\alpha t} & \left\{-\alpha + \exp\left(-\beta \sum_{i=1}^{n} (x_i + 2l_i)\right)\right. \\ & \left.\times \left[\alpha + \beta^2 \sum_{j,k=1}^{n} a_{jk}(x) - \beta \sum_{j=1}^{n} a_j(x)\right]\right\} \\ \geqslant Me^{-\alpha t} & \left\{-\alpha + \beta \exp\left(-\beta \sum_{i=1}^{n} (x_i + 2l_i)\right)\left[\beta a n - \sum_{j=1}^{n} a_j(x)\right]\right\}. \end{aligned}$$

Choose $\beta > 0$ large enough that $\beta > \sum_{j=1}^{n} a_j(x)/an$ for $x \in \omega$, and let $\sigma > 0$ be such that $\beta a n - \sum_{j=1}^{n} a_j(x) \geqslant \sigma$ for $x \in \omega$. Then

$$\Psi_t - L\Psi \geqslant Me^{-\alpha t}\left\{-\alpha + \beta\sigma \exp\left(-\beta \sum_{i=1}^{n} (x_i + 2l_i)\right)\right.$$
$$\geqslant Me^{-\alpha t}\{-\alpha + \beta\sigma e^{-3\beta\Sigma l_i}\},$$

which can be made positive by choosing $\alpha > 0$ sufficiently small. Now $\Psi|_{\partial\omega\times[0,\infty)} > 0$ and $\Psi|_{\omega\times\{0\}} \geqslant \text{const}\, M$, so M can be chosen large enough that the conditions of Lemma 1 are satisfied. With such a choice for M, Ψ is a barrier function for W^2, whence it follows that $W^2 \to 0$ as $t \to \infty$, uniformly in $x \in \omega$.

In a similar manner one can show the existence of positive constants α, β, and M independent of T and ε such that

$$\varepsilon M\left[1 - \exp\left(-\beta \sum_{i=1}^{n} (x_i + 2l_i)\right)\right]$$

is a barrier function for W^1 on $\omega \times [T, \infty)$. Thus for every $\varepsilon > 0$ there is a $T = T(\varepsilon)$ such that $|W^1| \leqslant M\varepsilon$ for $t \geqslant T$, and the theorem follows.

2. Reversible Polymer Formation

By a suitable choice of scales, we can assume that the system of equations governing this case has the form

$$-u_t + \Delta u = f(x, t, u, v), \qquad u|_\Gamma = \phi(x, t), \tag{3}$$
$$-\gamma v_t + \theta\,\Delta v = -f(x, t, u, v), \qquad v|_\Gamma = \psi(x, t), \tag{4}$$

where $\gamma > 0$, $\theta > 0$ are constants and f is a polynomial in u and v with coefficients which are continuous, bounded functions for $(x, t) \in \Omega$. On physical grounds we assume that f vanishes when both u and v vanish, that f is continuously differentiable in t, and that $f_u \geqslant 0$, $f_v \leqslant 0$ for $u, v \geqslant 0$. We assume that ϕ, ψ are continuously differentiable in t and nonnegtive. The existence of solutions of (3)–(4) may be established by standard arguments.

We first prove some results dealing with the Dirichlet problem obtained by dropping the time-derivative terms from (3)–(4).

LEMMA 4. *Let U, V be continuous functions on $\bar{\omega}$ satisfying*

$$\Delta U - aU + \beta V \geqslant 0, \tag{5}$$
$$\theta\,\Delta V + aU - \beta V \geqslant 0 \tag{6}$$

in ω and $U, V \leqslant 0$ on $\partial\omega$, where a, β are positive and $\theta > 0$ is a constant. Then $U, V \leqslant 0$ throughout ω.

Proof. By addition we have that $\Delta[U+\theta V] \geqslant 0$ in ω, $U+\theta V \leqslant 0$ on $\partial\omega$. From the ordinary maximum principle, $U+\theta V \leqslant 0$ throughout ω. Since $\theta > 0$, we conclude that if either U or V is positive at a point of ω, the other is negative there. Suppose, contrary to the conclusion of the theorem, that either U or V has a positive maximum at some point $\bar{x} \in \omega$, say $U(\bar{x}) > 0$ without loss of generality. Since $\Delta U(\bar{x}) \leqslant 0$ and $V(\bar{x}) < 0$, (5) is contradicted. Hence the theorem.

COROLLARY. *Let U, V be solutions of $\Delta U - \alpha U + \beta V = 0$, $\theta \Delta V + \alpha U - \beta V = 0$ with nonnegative boundary data. Then U and V are nonnegative. If the boundary data are nonnegative bounded functions of a parameter $t \in [0, T]$ (or $[0, \infty)$), then U and V are bounded uniformly on $\omega \times [0, T]$ (resp., $\omega \times [0, \infty)$).*

Proof. Nonnegativity follows by applying the theorem to $-U$, $-V$. Let M denote a bound on the boundary values of both U and V for $t \in [0, T]$. Since $\Delta M = 0$ we have as in the theorem that $U + \theta V \leqslant (1+\theta) M$. From the nonnegativity of U and V it follows that both are bounded.

COROLLARY. *Let U and V solve*

$$\Delta U = f(x, t, U, V), \qquad U|_{\partial\omega} = \phi(x, t), \tag{7}$$

$$\theta \Delta V = -f(x, t, U, V), \qquad V|_{\partial\omega} = \psi(x, t), \tag{8}$$

for $t > 0$. Assume that ϕ and ψ are bounded and nonnegative on Γ. Then U, V are nonnegative bounded functions.

Proof. Since f vanishes when both U and V are zero, we may write

$$\begin{aligned} f(x, t, U, V) &= f(x, t, U, V) - f(x, t, 0, 0) \\ &= f_u(x, t, U^*, V)\, U + f_v(x, t, 0, V^*)\, V, \end{aligned}$$

where $0 \leqslant U^* \leqslant U$, $0 \leqslant V^* \leqslant V$. The preceding corollary now yields the result.

LEMMA 5. *Let $U(x, t)$, $V(x, t)$ be a twice continuously differentiable (in x) solution in ω of (7)–(8). Then U, V are continuously differentiable in t and U_t, V_t satisfy*

$$\Delta U_t = f_t(x, t, U, V) + f_U(x, t, U, V)\, U_t + f_V(x, t, U, V)\, V_t,$$

$$\theta \Delta V_t = -f_t(x, t, U, V) - f_U(x, t, U, V)\, U_t - f_V(x, t, U, V)\, V_t,$$

$$U_t|_{\partial\omega} = \phi_t, \qquad V_t|_{\partial\omega} = \psi_t.$$

Proof. Let $W(x,t) \equiv U(x,t) + \theta V(x,t)$. Since $\Delta W = 0$, $W|_{\partial\omega} = \phi + \theta\psi$, we have by Lemma 2 that W is continuously differentiable in t and satisfies $\Delta W_t = 0$, $W_t|_{\partial\omega} = \phi_t + \theta\psi_t$. Upon substitution, we get that U satisfies

$$\Delta U = f(x, t, U, \theta^{-1}[W(x,t) - U]), \qquad U|_{\partial\omega} = \phi.$$

Noting that $(\partial/\partial U) f(x, t, U, \theta^{-1}[W(x,t) - U]) = f_U - \theta^{-1} f_V \geqslant 0$, we see that the hypotheses of Lemma 2 are again satisfied and thus that

$$\begin{aligned}\Delta U_t &= f_t(x,t,U,\theta^{-1}[W(x,t)-U]) + \theta^{-1} f_V(x,t,U,\theta^{-1}[W(x,t)-U])\, W_t(x,t) \\ &\quad + f_U(x,t,U,\theta^{-1}[W(x,t)-U])\, U_t - \theta^{-1} f_V(x,t,U,\theta^{-1}[W(x,t)-U])\, U_t \\ &= f_t(x,t,U,V) + f_U(x,t,U,V)\, U_t + f_V(x,t,U,V)\, V_t,\end{aligned}$$

$$U_t|_{\partial\omega} = \phi_t.$$

The rest of the lemma now follows easily.

LEMMA 6. *Assume that f satisfies, in addition to our standing hypotheses,*

(i) *the coefficients of the terms of f are bounded in Ω, and*

(ii) *the time derivative of each coefficient tends to zero as $t \to \infty$, uniformly for $x \in \omega$.*

Assume that ϕ and ψ are bounded and that $\phi_t(x,t)$, $\psi_t(x,t) \to 0$ uniformly in $x \in \partial\omega$ as $t \to \omega$. Let $U(x,t)$, $V(x,t)$ be a nonnegative solution pair for the Dirichlet problem (7)–(8). Then $U_t(x,t)$, $V_t(x,t) \to 0$ as $t \to \infty$, uniformly in $x \in \omega$.

Proof. Again letting $W = U + \theta V$, we have that $\Delta W = 0$, $W|_{\partial\omega} = \phi + \theta\psi$ and thus $W_t \to 0$ uniformly in $x \in \omega$ as $t \to \infty$ by Lemma 3. As before, U satisfies $\Delta U = f(x, t, U, \theta^{-1}[W(x,t) - U])$; we have

$$\begin{aligned}&\frac{\partial}{\partial t} f(x,t,U,\theta^{-1}[W(x,t)-U]) \\ &\qquad = f_t(x,t,U,\theta^{-1}[W(x,t)-U]) + \theta^{-1} f_V(x,t,U,\theta^{-1}[W(x,t)-U])\, W_t.\end{aligned}$$

Since V and U, and hence W, are bounded by the second corollry to Lemma 4, it follows from Lemma 3 that $U_t(x,t) \to 0$ as $t \to \infty$, uniformly in $x \in \omega$, and hence that $V_t(x,t) \to 0$ also.

The final tool needed is the following lemma, which is a special case of a result in [5, p. 190].

LEMMA 7. *Let c and e be bounded nonnegative functions, and let b and d be bounded. Let u, v satisfy*

$$-u_t + \Delta u + bu + cv \leqslant 0,$$
$$-\gamma v_t + \theta \Delta v + dv + eu \leqslant 0$$

in Ω_T with $u, v \geqslant 0$ on Γ_T. Then $u(x, t)$, $v(x, t) \geqslant 0$ throughout Ω_T.

We are now ready to prove

THEOREM 2. *Let the hypotheses of Lemma 6 hold. Let u, v be a bounded solution pair for the problem (3)–(4) and let U, V be the solution pair for (7)–(8). Then $u(x, t) - U(x, t) \to 0$, $v(x, t) - V(x, t) \to 0$ uniformly in $x \in \omega$ as $t \to \infty$.*

Proof. Let $y = u - U$, $z = v - V$; then

$$\begin{aligned} -y_t + \Delta y &= f(x, t, u, v) - f(x, t, U, V) + U_t \\ &= f_u(x, t, u^*, v^*)\, y + f_v(x, t, U^*, V^*)\, z + U_t \end{aligned} \tag{9}$$

by the mean value theorem, where u^*, U^* lie between u and U and v^*, V^* lie between v and V. Similarly, z satisfies

$$-\gamma z_t + \theta \Delta z = -f_u(x, t, \bar{u}, \bar{v})\, y - f_v(x, t, \bar{U}, \bar{V})\, z + \gamma V_t. \tag{10}$$

Clearly,

$$y|_{\partial\omega \times [0,\infty)} = 0 = z|_{\partial\omega \times [0,\infty)},$$

and both y and z are bounded at $t = 0$.

Note that the coefficients of y and z on the right hand sides of (9)–(10) are actually continuous in Ω. To see this, let $b(x, t)\, u^n v^m$ be any term in f; then

$$\begin{aligned} &b(x, t)\, u^n v^m - b(x, t)\, U^n V^m \\ &\quad = b(x, t)\{[u^n - U^n]\, v^m + U^n[v^m - V^m]\} \\ &\quad = b(x, t)\, p_1(u, U)\, v^m(u - U) + b(x, t)\, p_2(v, V)\, U^n(U - V), \end{aligned}$$

where the p_i are polynomials in the indicated arguments. The claim that the coefficients on the right hand side of (9)–(10) are continuous and bounded now follows. It also follows that we can take $\bar{u} = u^*$, $\bar{v} = v^*$, $\bar{U} = U^*$, $\bar{V} = V^*$.

The system (9)–(10) thus has the form studied in [3, p. 352] except for the presence of the terms U_t and γV_t on the right hand sides of (9) and (10), respectively, and the presence of $\gamma \not\equiv 1$ on the left in (10). However, Kahane's proof that y and z approach zero in the L_p $(p \geqslant 1)$ sense as $t \to \infty$ extends easily to the present case since for any function $\phi_1(x)$ continuous on $\bar{\omega}$ we have that

$$\int_\omega \phi_1(x)[U_t(x, t) + \gamma V_t(x, t)]\, dx \to 0$$

as $t \to \infty$ as an easy consequence of Lemma 6. Finally, from Lemma 3.1 of [3] we now get the conclusion of the theorem.

The weakness of the preceding result is the requirement that u and v be bounded. We have been unable to find satisfactory general conditions on f which will guarantee boundedness. However, the following result covers the case of greatest physical interest, namely, constant reaction rates.

COROLLARY. *In addition to the hypotheses of Theorem* 2, *assume that* f *has the form*

$$f(x, t, u, v) = g(x, t)[u^n - \delta v],$$

where δ *is a constant. Then* u *and* v *are bounded and therefore* $u(x, t) - U(x, t) \to 0$, $v(x, t) - V(x, t) \to 0$ *as* $t \to \infty$, *uniformly in* $x \in \omega$.

Proof. For arbitrary constants M and N we have that

$$\begin{aligned} &-(u - M)_t + \Delta(u - M) - g(x, t)(u^{n-1} + Mu^{n-2} + \cdots + M^{n-1})(u - M) \\ &\qquad + g(x, t)\,\delta(V - N) = g(x, t)[M^n - \delta N], \\ &-\gamma(V - N)_t + \Delta(v - N) + g(x, t)(u^{n-1} + Mu^{n-2} + \cdots + M^{n-1})(u - M) \\ &\qquad - g(x, t)\,\delta(V - N) = -g(x, t)[M^n - \delta N]. \end{aligned}$$

Both right hand sides vanish provided we choose $N = M^n/\delta$. If we further insist that $(u - M)|_\Gamma \leqslant 0$, $(v - N)|_\Gamma \leqslant 0$, then Lemma 4 can be invoked to conclude that $u \leqslant M$, $v \leqslant N$ throughout Ω.

3. IRREVERSIBLE REACTION OF TWO CHEMICALS

After normalization, the equations of the introduction describing this sort of reaction can be written as

$$\begin{aligned} -u_t + \Delta u &= u^n v^m, &\qquad u|_\Gamma &= \phi, \\ -\gamma v_t + \theta\,\Delta v &= u^n v^m, &\qquad v|_\Gamma &= \psi. \end{aligned}$$

Slight adaptation of the "monotone scheme" approach of [4], coupled with the results of Section 1 for the single equation, readily yields the following result, whose proof is therefore omitted.

THEOREM 3. *Let $f(x, t, u, v)$ be continuous in $\bar{\Omega}$ and continuously differentiable in t, u, and v there, with $f_u \geqslant 0, f_v \geqslant 0$ for $u, v \geqslant 0$. Let f, f_u, and f_v be bounded for $(x, t) \in \omega$ and u, v bounded, and let $f_t(x, t, u, v) \to 0$ as $t \to \infty$ uniformly in x, u, v for $x \in \omega$ and u, v bounded. Let $\phi, \psi \geqslant 0$ be bounded and let $\phi_t(x, t)$, $\psi_t(x, t) \to 0$ as $t \to \infty$, uniformly for $x \in \partial\omega$. Let $u(x, t)$, $v(x, t)$ solve*

$$-u_t + \Delta u = f(x, t, u, v), \qquad u|_\Gamma = \phi, \tag{11}$$

$$-\gamma v_t + \theta \Delta v = f(x, t, u, v), \qquad v|_\Gamma = \psi. \tag{12}$$

Let $U(x, t)$, $V(x, t)$ be the unique solution of the Dirichlet problem

$$\Delta U = f(x, t, U, V), \qquad U|_{\partial\omega} = \phi,$$

$$\theta \Delta V = f(x, t, U, V), \qquad V|_{\partial\omega} = \psi,$$

for $t \geqslant 0$. Then $u(x, t) - U(x, t) \to 0$, $v(x, t) - V(x, t) \to 0$ uniformly in $x \in \omega$ as $t \to \infty$.

This result can be established for more general elliptic operators, and for distinct elliptic operators in (11) and (12).

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